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Analog Circuits for Energy and Fuel Optimal Control of Linear Discrete Systems*

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Summary-- The purpose of this paper is to explore the possibility of using analog networks for solving minimum fuel and minimum energy problems for linear discrete systems. The necessary theorems for circuits consisting of nonlinear resistors, current and voltage sources, and transformers are developed. Physical components are substantially accounted for and a method of implementation for simple systems is indicated.

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INTRODUCTION

In general, the implementation of optimal control strategies requires the use of fast digital computers. As a result, optimal control of fairly simple and inexpensive systems is usually unfeasible for economic reasons. A possible way out of this impasse may be found in the form of passive analog networks which solve the optimal control problem. As the reader probably knows, the idea of using analog networks for solving programming problems is not entirely new.[1, 2] However, to the best of the authors' knowledge, the relevant theorems for networks consisting of transformers, resistive elements and several mixed sources have not been proved. Consequently, the first part of this paper is devoted to writing network equations in a compatible vector form and to proving the theorems on which the optimization analogs are to be based.

The main purpose of this paper is to explore to some degree the possibilities of analog networks for solving two optimization problems: optimal fuel and optimal energy for discrete systems with linear plants. Since ideal networks are not realizable, some thought is devoted to the errors introduced by the nonideal characteristics of physical components and to a possible physical implementation of such networks. Generally speaking, it appears that such networks are best suited for the control of simple, mass produced systems.

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DESCRIPTION OF THE SYSTEM

Consider an n-th order, time invariant, discrete system described by the linear vector difference equation

$$\underline{x}_{j+1} = \underline{C}\,\underline{x}_j + u_{j+1}\underline{d} \tag{1}$$

where $\underline{x}_j \in E^n$ is the state of the system at the j-th sampling instant, $j = 0, 1, 2, \ldots, u_j$ is a bounded constant scalar which is the system input during the j-th sampling period, $\underline{C}$ is a constant, real nonsingular $n \times n$ matrix, and $\underline{d} \in E^n$ is a constant vector. It will be assumed that the system (1) is completely controllable, i.e., that the vectors $\underline{d}, \underline{C}\underline{d}, \ldots, \underline{C}^{(n-1)}\underline{d}$ are linearly independent.

Let $\underline{x}_0$ be the initial state of the system (1) and let $(u_1, u_2, \ldots, u_N)$ be a sequence of controls. Then, from (1), the corresponding state at the N-th sampling instant is

$$\underline{x}_N = \underline{C}^N \underline{x}_0 + \sum_{j=1}^{N} \underline{C}^{N-j}\,\underline{d}u_j \ . \tag{2}$$

Eq. (2) may now be rewritten as follows:

$$(\underline{C}^{-N}\underline{x}_N - \underline{x}_0) = \sum_{j=1}^{N} \underline{C}^{-j}\underline{d}u_j \ . \tag{3}$$

As a result, the two-point boundary-value problem--"given $\underline{x}_0$ and $\underline{x}_N$ find a sequence of controls $(u_1, \ldots, u_N)$ such that (2) is satisfied"--transforms into the problem of finding a sequency of controls $(u_1, \ldots, u_N)$ such that

$$\underline{R}\,\underline{u} = \underline{z}_N \tag{4}$$

where $\underline{z}_N = (C^{-N}\underline{x}_N - \underline{x}_0)$ is the given target state, $\underline{u} = \mathrm{col}(u_1, \ldots, u_N)$ is the required sequence of controls represented as a vector in E^N, and R is a $n \times N$ matrix whose j-th column $\underline{r}_j = \underline{C}^{-j}\underline{d}$, $j = 1, 2, \ldots, N$.

Definition: Let S_N be the set of all the target states $\underline{z}_N$ which can be reached from an arbitrary initial state $\underline{x}_0$ in N sampling periods with controls $|u_j| \leq 1$, $j = 1, 2, \ldots, N$, i.e.,

$$S_N = \left\{ \underline{z}_N : \underline{z}_N = \underline{Ru},\ \underline{u} = \mathrm{col}(u_1, \ldots, u_N),\ |u_j| \leq 1,\ j = 1, 2, \ldots N \right\}. \tag{5}$$

Definition: For a given target state $\underline{z}_N \epsilon S_N$, let U be the set of all the bounded controls $\underline{u}$ which satisfy (4), i.e.,

$$U = \left\{ \underline{u}: \underline{Ru} = \underline{z}_N,\ \underline{u} = \mathrm{col}(u_1, \ldots, u_N),\ |u_j| \leq 1,\ j = 1, 2, \ldots, N \right\}. \tag{6}$$

STATEMENT OF THE PROBLEM

For the system (1), given the integer N and a target state $\underline{z}_N \epsilon S_N$, find a control $\underline{u}^e \epsilon U$ and a control $\underline{u}^f \epsilon U$ such that

(i) $$J_e(\underline{u}^e) = \min_{u \epsilon U} J_e(\underline{u}) \quad \text{(minimum energy)} \tag{7}$$

where $$J_e(\underline{u}) = \sum_{j=1}^{N} u_j^2 \tag{8}$$

(ii) $$J_f(\underline{u}^f) = \min_{u \epsilon U} J_f(\underline{u}) \quad \text{(minimum fuel)} \tag{9}$$

where $$J_f(\underline{u}) = \sum_{j=1}^{N} |u_j| . \tag{10}$$

Remark: Since J_e is strictly convex and U is also convex, there is a unique control $\underline{u}_e$ which satisfies (7). However, J_f is convex, but not strictly convex, and hence the solution of (ii) above need not be unique. In fact, there may be an infinite number of solutions. It will be observed that (ii) is a problem in linear programming.[3]

It will now be shown that there exist ideal component circuits which solve the problems (i) and (ii) exactly and that circuits using nonideal components may approach the desired solution arbitrarily closely. However, first it will be necessary to extend and strengthen two theorems due to Duffin[4] and Millar.[5]

DESCRIPTION OF THE NETWORK $\mathcal{N}$

Consider a network $\mathcal{N}$ consisting of h independent current sources, i_{csj}, $j = 1, 2, \ldots, h$, k independent voltage sources, v_{vsj}, $j = 1, 2, \ldots, k$, m nonlinear resistros, and p ideal transformers having a total of n windings. The network $\mathcal{N}$ will be assumed to satisfy the following conditions:

(1) For $j = 1, 2, \ldots, m$, the voltage v_{rj} across the j-th nonlinear resistor is related to the current i_{rj} in this resistor by the equation

$$v_{rj} = g_j(i_{rj}), \quad j = 1, 2, \ldots, m, \tag{11}$$

where the $g_j(\cdot)$ are continuous, monotonic increasing functions.

(2) The network $\mathcal{N}$ obeys the Kirchhoff current and voltage laws (KCL, KVL) independently of the numerical values of the current and voltage sources. In particular, this implies that there do not exist in $\mathcal{N}$ any loops consisting of voltage sources only, or cut sets consisting of current sources only.

(3) The ideal transformers introduce some further restrictions on $\mathcal{N}$. Thus, let $\underline{v}_t$, $\underline{i}_t \in E^n$ be vectors whose elements are the n transformer coil voltages and currents. Following Carlin

and Giordano,[6] one may define flux meshes and, analogously, flux nodes for the transformers and then write down the $n \times q$ coil-turns-flux-mesh incidence matrix $\underline{M}$ and the $n \times s$ coil-turn-flux-node incidence matrix $\underline{N}$, where q is the number of flux meshes and s is the number of flux nodes. In terms of these matrices, the transformer relations are:

$$\underline{M}' \underline{i}_t = \underline{0} \tag{12}$$

$$\underline{N}' \underline{v}_t = \underline{0} \tag{13}$$

$$(\underline{v}_t, \underline{i}_t) = \sum_{j=1}^{n} v_{tj} i_{tj} = 0 \,. \tag{14}$$

The rank of $\underline{M}$ is q and the rank of $\underline{N}$ is s and both of these must be strictly less than n, the number of coils. This imposes a restriction on the manner of interconnecting various windings. Furthermore, (12) and (13) impose a restriction on the manner of connecting current and voltage sources to the transformer windings.

KIRCHHOFF'S CURRENT AND VOLTAGE LAWS FOR $\mathcal{N}$

In order to be able to show that a specific network $\mathcal{N}$ is capable of solving one or the other of the problems (7), (8), it will be necessary to write KCL and KVL in a suitable form, and hence to deduce some further relations for the various branch current and voltages.

Let $\underline{i}_{cs}$, $\underline{i}_r$, $\underline{i}_{vs}$, $\underline{i}_t$ be vectors whose elements are the current source, resistor, voltage source and transformer winding currents, and let $\underline{v}_{cs}$, $\underline{v}_r$, $\underline{v}_{vs}$, $\underline{v}_t$ be vectors whose elements are the voltages corresponding to these currents. Furthermore, let $\underline{i}_b = (\underline{i}_{cs}, \underline{i}_r, \underline{i}_{vs}, \underline{i}_t)'$, $\underline{v}_b = (\underline{v}_{cs}, \underline{v}_r, \underline{v}_{vs}, \underline{v}_t)'$, and let $\underline{i}_{bcs}$, $\underline{i}_{br}$, $\underline{i}_{bvs}$, $\underline{i}_{bt}$, $\underline{v}_{bcs}$, $\underline{v}_{br}$, $\underline{v}_{bvs}$, $\underline{v}_{bt}$ be vectors of the same dimension as $\underline{i}_b$ and $\underline{v}_b$ and obtained from these by setting to zero all but the indicated components; thus $\underline{i}_{bcs} = (\underline{i}_{cs}, \underline{0}, \underline{0}, \underline{0})$, etc.

Now, considering each current source, voltage source, resistor and winding as a separate branch, let us construct a tree for $\mathcal{N}$ containing no current sources (if under the above hypothesis the circuit has more than one separate part, a tree is to be constructed for each one). Let $\underline{i}_\ell = (\underline{i}_{cs}, \underline{i}_{\ell 1}) \in E^u$ be a vector whose elements are the current source currents as well as the currents through the remaining chords (branches not in the tree). Then, the vector $\underline{i}_\ell$ specifies a unique set of fundamental loops and fundamental loop currents for $\mathcal{N}$. In terms of these,

$$\text{KCL is } \underline{i}_b = \underline{A}\,\underline{i}_\ell \tag{15}$$

and

$$\text{KVL is } \underline{A}'\underline{v}_b = \underline{0}. \tag{16}$$

where $\underline{A}$ is a $n \times u$ dimensional matrix of the form

$$\underline{A} = \left(\begin{array}{c|c} \underline{I} & \underline{0} \\ \hline \underline{A}_1 & \underline{A}_2 \end{array}\right).$$

Eq. (12) may now be used to bring in the restrictions due to the transformers. These may be cast into the form

$$\underline{i}_\ell = \underline{T}\,\underline{i}_f \tag{16a}$$

where $\underline{i}_f = (\underline{i}_{cs}, \underline{i}_{f1}) \in E^v$, $v \leq u$, $\underline{i}_{f1}$ being a vector whose elements are some of the elements of $\underline{i}_{\ell 1}$. The elements of $\underline{i}_f$ form a set of basic current variables which, once given, determine all the currents in the network $\mathcal{N}$ by means of the relation

$$\underline{i}_b = \underline{A}\,\underline{T}\,\underline{i}_f . \tag{17}$$

The $u \times v$ matrix $\underline{T}$ has the form

$$\underline{T} = \left(\begin{array}{c|c} \underline{I} & \underline{0} \\ \hline \underline{T}_1 & \underline{T}_2 \end{array}\right)$$

where the identity matrix has the same dimensions as the one in the matrix $\underline{A}$. Hence

$$\underline{T}'\underline{A}'\underline{v}_{bcs} = (\underline{v}_{cs}, \underline{0}) \tag{18}$$

where the vector in the right-hand-side of (18) is of demension v. Now combining (14) and (17) one obtains

$$(\underline{T}'\underline{A}'\underline{v}_{bt} ; \underline{i}_f) = \underline{0} \quad . \tag{19}$$

Since (19) must hold independently of the values of $\underline{i}_f$, the transformer voltage relations are, in an alternate form to (13),

$$\underline{T}'\underline{A}'\underline{v}_{bt} = \underline{0} \quad . \tag{20}$$

Now, let $\underline{D}$ be a $v \times v$ matrix of the form

$$\underline{D} = \left(\begin{array}{c|c} \underline{0} & \underline{0} \\ \hline \underline{0} & \underline{I} \end{array}\right)$$

where $\underline{I}$ is a $(v - h) \times (v - h)$ identity matrix. Then, combining (16), (18) and (20), one obtains

$$\underline{D}\underline{T}'\underline{A}'\underline{v}_b = \underline{D}\underline{T}'\underline{A}'(\underline{v}_{bvs} + \underline{v}_{br}) = \underline{0} \quad . \tag{21}$$

Eqs. (17) and (21) will be called the Kirchhoff law derived current and voltage constraints, (KCLD and KVLD).

THE FUNCTIONALS $J(\underline{i}_b^*)$ AND $J(\underline{v}_b^*)$

Let the actual current and voltage vectors of the network $\mathcal{N}$ be $\underline{i}_b^o$ and $\underline{v}_b^o$, i.e., they satisfy all the equations above. When all the restrictions imposed by the network on the branch currents and voltages are relinquished, except for (11), while the sources are retained, it may be assumed that the branch current vector has the arbitrary value $\underline{i}_b^* = (\underline{i}_{cs}, \underline{i}_r^*, \underline{i}_{vs}^*, \underline{i}_t^*)$ and that the branch voltage vector has the arbitrary value $\underline{v}_b^* = (v_{cs}^*, \underline{v}_r^*, \underline{v}_{vs}, \underline{v}_t^*)$. It is understood of course, that $\underline{v}_r^*$ and $\underline{i}_r^*$ are related by (11). It is now possible to associate with the network $\mathcal{N}$ the functionals

$$J_p(\underline{i}_b^*) = \int_o^{\underline{i}_b^*} (\underline{v}_{br} + \underline{v}_{bvs}) \cdot d\underline{i}_b \tag{22}$$

$$J_d(\underline{v}_b^*) = \int_o^{\underline{v}_b^*} (\underline{i}_{br} + \underline{i}_{bcs}) \cdot d\underline{v}_b \;. \tag{23}$$

It will be observed that the above line integrals are independent of the path of integration and that they are convex in i_b^* and v_b^* respectively. The above two functionals, in a sense, bear a primal-dual relationship to each other.

<u>THEOREM</u>: Subject to the constraint of KCL and the transformers, i.e., $\underline{i}_b^* = \underline{A}\,\underline{T}\,\underline{i}_f^*$ (KCLD, Eq. (17)), the functional

$$J_p(\underline{i}_b^*) = \int_o^{\underline{i}_b^*} (\underline{v}_{br} + \underline{v}_{bvs}) \cdot d\underline{i}_b,$$

associated with the network $\mathcal{N}$, assumes an absolute minimum at the actual current distribution of $\mathcal{N}$, i.e., for $\underline{i}_b^* = \underline{i}_b^o$.

<u>PROOF</u>: Since J_p is convex in i_b^* and the constraints are also convex, it follows that if J_p has a constrained stationary

value, this value must be a minimum and it is unique. The value of i_b^* at which this minimum is achieved, however, need not be unique. Now,

$$dJ_p(\underline{i}_b^o) = (\nabla J_p(\underline{i}_b^o),\ d\underline{i}_b^*) \tag{24}$$

where

$$\nabla J_p(\underline{i}_b^o) = (\partial J_p(\underline{i}_b)/\partial i_{b1}, \ldots, \partial J_p(\underline{i}_b)/\partial i_{bN}) \Big|_{\underline{i}_b = \underline{i}_b^o},\ N = (h+k+m+n),$$

and

$$d\underline{i}_b^* = \underline{A}\,\underline{T}\,\underline{D}\,d\underline{i}_f^* ,$$

since $d\underline{i}_f^* = (0,\ d\underline{i}_{f1}^*)$. Combining (11), (24), and (21), one obtains:

$$dJ_p(\underline{i}_b^o) = (\underline{D}\,\underline{T}'\underline{A}'(\underline{v}_{br}^o + \underline{v}_{bvs}),\ d\underline{i}_f^*) = 0, \quad \text{for all } d\underline{i}_f^* \neq 0.$$

Hence $J_p(\underline{i}_b^*)$ has a constrained stationary value at $\underline{i}_b^* = \underline{i}_b^o$ and this value must be the constrained minimum value of J_p. QED.

After rewriting KCL and KVL in a suitable form, and proceeding essentially as before, it becomes possible to prove the following "dual" theorem.

DUAL THEOREM: Subject to the constraints of KVL and the transformers (analogous in form to (21)), the functional

$$J_d(\underline{v}_b^*) = \int_o^{\underline{v}_b^*} (\underline{i}_{br} + \underline{i}_{bcs}) \cdot d\underline{v}_b$$

associated with the network $\mathcal{N}$, assumes an absolute minimum at the actual voltage distribution of $\mathcal{N}$, i.e., for $\underline{v}_b^* = \underline{v}_b^o$.

CURRENT ANALOG FOR THE MINIMUM ENERGY PROBLEM

Consider the network shown in Fig. 1. It will now be shown that this network can be used to solve the minimum energy problem (7). The transformers in this network are ideal, with turn ratios $1:r_{jk}$, $j = 1, 2, \ldots, n$, and $k = 1, 2, \ldots, N$. The sources are dc current sources of magnitude z_N^j, $j = 1, 2, \ldots, n$. The N nonlinear resistors have identical terminal characteristics which are shown in Fig. 2(a) and a realization for them is shown in Fig. 2(b). The incremental resistance of these elements is equal to two for $|i_{rk}| < 1$ and it is equal to s for $|i_{rk}| > 1$, $k = 1, 2, \ldots, N$. Let us assume that the incremental resistance s, which is introduced to exhibit the effect of using nonideal circuits, may be used as a parameter for this network. The network will be denoted by $\mathcal{N}^{\ell}_{cs}$. Then the functional (22) which may be associated with $\mathcal{N}^{\ell}_{cs}$ will also depend parametrically on s, and, since there are no voltage sources in $\mathcal{N}^{\ell}_{cs}$, it assumes the simple form

$$J_{ps}(\underline{i}_{br}^{*}) = \int_{o}^{\underline{i}_{br}^{*}} \underline{v}_{br} \cdot d\underline{i}_{br}$$

$$= \int_{o}^{\underline{i}_{r}^{*}} \underline{v}_{r} \cdot d\underline{i}_{r}$$

$$= \sum_{k \epsilon M} i_{rk}^{*2} + \sum_{k \epsilon M^c} \left(2|i_{rk}^{*}| - 1 + \frac{1}{2} (|\underline{i}_{rk}^{*}| - 1)^2 s \right) \quad (25)$$

where $\underline{i}_r^{*} = (i_{r1}^{*}, \ldots, i_{rN}^{*})$, $M \subset \{1, 2, \ldots, N\}$ is an index set such that if $k \epsilon M$ then $|i_{rk}| \leq 1$, and M^c is the complement of M in $\{1, 2, \ldots, N\}$.

The KCL and the transformers impose the following relation on the resistor currents:

$$\underline{z}_N = \underline{R}\underline{i}_r \tag{26}$$

where $\underline{z}_N = (z_N^1, \ldots, z_N^n)$ is the current source vector whose value will be assumed to be identical with that of the target state for (7) and $\underline{R} = (r_{jk})$ is a $n \times N$ matrix whose components are the transformer ratios. Again, it will be assumed that the matrices $\underline{R}$ in (26) and in (6) are identical.

Now, let $U_1 \subset E^N$ be the set of all vectors $\underline{i}_r$ which satisfy (26), and let $\underline{i}_{rs}^o$ be the actual resistor current vector for $\mathcal{N}_{cs}$. Then, since from (6) $U \subset U_1$,

$$J_{ps}(\underline{i}_{brs}^o) = \min_{\underline{i}_r \epsilon U_1} J_{ps}(\underline{i}_{br}) \leq \min_{\underline{i}_r \epsilon U} J_{ps}(\underline{i}_{br}) \ . \tag{27}$$

However, it is seen from (25) and (8) that

$$J_{ps}(\underline{i}_{br}) \equiv J_e(\underline{i}_r) \ \text{ for all } \underline{i}_r \epsilon U \ . \tag{28}$$

Since by assumption (7) has a solution in U, it follows that for all $s > 0$

$$J_{ps}(\underline{i}_{brs}^o) \leq \min_{\underline{i}_r \epsilon U} J_e(\underline{i}_r) < \infty \ . \tag{29}$$

Since (29) must hold when $s \rightarrow \infty$, it is clear that $\underline{i}_{r\infty}^o \epsilon U$, that

$$J_{p\infty}(\underline{i}_{br}^o) = \min_{\underline{u} \epsilon U} J_e(\underline{u}) \tag{30}$$

and that $\underline{u}_e = \underline{i}_{r\infty}^o$ is the required solution for the minimum energy problem. Thus, for $s = \infty$, the circuit solves the minimum energy proglem by a current analogy.

VOLTAGE ANALOG FOR THE MINIMUM ENERGY PROBLEM

Now consider the network shown in Fig. 3. The ideal transformers have turns ratios $1:r_{jk}$, $j = 1, 2, \ldots, n$, $k = 1, 2, \ldots, N$, the ideal sources z_N^j, $j = 1, 2, \ldots, n$, are voltage sources, and the nonlinear resistors have characteristics shown in Fig. 4(a) and their realization is shown in Fig. 4(b). Again, the network, which will be denoted by $\mathcal{N}_{vs}^{\ell}$, depends parametrically upon the incremental resistance s. For $\mathcal{N}_{vs}^{\ell}$,

$$J_{ds}(\underline{v}_{br}^{*}) = \int_{o}^{\underline{v}_{b\ell}^{*}} \underline{i}_{br} \cdot d\underline{v}_{br}$$

$$= \int_{o}^{\underline{v}_{r}^{*}} \underline{i}_{r} \cdot d\underline{v}_{r}$$

$$= \sum_{k \in M} v_{rk}^{*2} + \sum_{k \in M^c} \left(2 |v_{rk}^{*}| - 1 + \frac{1}{2} (|v_{rk}^{*}| - 1)^2 s \right) \quad (31)$$

where the set M is defined in the same manner as before.

The KVL and transformers impose the following relation on the voltages across the nonlinear resistors:

$$\underline{z}_N = \underline{R}\underline{v}_r \quad (32)$$

where $\underline{z}_N = (z_N^1, \ldots, z_N^n)$ is the voltage source vector whose value is assumed to be identical to that of the target state for (7) and $\underline{R} = (r_{jk})$ is a $n \times N$ matrix, whose components are the turns ratios, and which will again be assumed to be identical with the matrix R in (6). By an argument identical to the one for the current analog, it follows from the dual theorem that for $s = \infty$ the network $\mathcal{N}_{vs}^{\ell}$ solves the minimum energy problem with the optimal control $\underline{u}_e = \underline{v}_r^{o}$.

CURRENT ANALOG FOR THE MINIMUM FUEL PROBLEM

In order to solve the minimum fuel problem (9), the network shown in Fig. 1 is used again, but with different resistive elements. The terminal characteristics of these resistive elements, which are again all identical, are shown in Fig. 5(a) and their realization in Fig. 5(b). As before, the network, which will be denoted by η^f_{cs}, depends on a parameter s. It will be observed that (26) represents the KCL and transformer constraints for this network as well. For η^f_{cs}

$$J_{ps}(\underline{i}_{br}^{*}) = \frac{1}{2} \sum_{k\epsilon M_1} i^2_{rk}\, s + \frac{1}{2} \sum_{k\epsilon M_2} \Big[(2|i_{rk}| - 1/s) + (|i_{rk}| - 1/s)^2\, s/(s-1)^2 \Big]$$

$$+ \frac{1}{2} \sum_{k\epsilon M_3} (2(|i_{rk}| - 1)/(s-1) + (|i_{rk}| - 1)^2 s) \qquad (33)$$

where M_1, M_2, M_3, $\subset \{1, 2, \ldots, N\}$ are index sets defined as follows: $k\epsilon M_1$ if $|i_{rk} < 1/s$, $k\epsilon M_2$ if $1/s \leq |i_{rk}| \leq 1$, and $k\epsilon M_3$ if $|i_{rk}| > 1$. It may now be shown in the same way as for the minimum energy problem that when $s = \infty$ the index sets M_1 and M_3 must become empty, i.e., that $i_{r\,\infty}^{o}\, \epsilon U$ and that

$$J_{p\,\infty}(\underline{i}_{br}^{*}) \equiv J_f(\underline{i}_r^{*}) \text{ for all } \underline{i}_r^{*} \epsilon U .$$

Hence $\underline{u}_f = \underline{i}_{r\,\infty}^{o}$ is a solution of the problem (9). Note that when $s = \infty$ the network η^f_{cs} becomes indeterminate in the same manner as the linear program (9) and may be capable of an infinitude of solutions, each of which is equally good.

VOLTAGE ANALOG FOR THE MINIMUM FUEL PROBLEM

When the resistors shown in Fig. 6(a) (realization in Fig. 6(b)) are substituted for the resistors in Fig. 3, one obtains a new network η^f_{vs} which can be used to solve the minimum fuel problem by means of a voltage analogy to the control u_f. The restrictions due to KVL and the transformers on the resistor voltages are still given by (27), while the associated functional J_{ds} becomes

$$J_{ds}(\underline{v}_{br}{}^*) = \frac{1}{2} \sum_{k\epsilon M_1} v_{rk}^2 \, s + \frac{1}{2} \sum_{k\epsilon M_2} \left[(2|v_{rk}| - \frac{1}{s}) + (|v_{rk}| - \frac{1}{s})^2 \frac{s}{(s-1)^2} \right]$$

$$+ \frac{1}{2} \sum_{k\epsilon M_3} \left[\frac{2(|v_{rk}| - 1)}{s - 1} + (|v_{rk}| - 1)^2 \, s \right] \qquad (34)$$

where the sets M_1, M_2, and M_3 are defined in the same manner as before.

Again, when $s \to \infty$, the network η^f_{vs} solves the minimum fuel problem (9), with $\underline{u}_f = \underline{v}_{r\,\infty}^{\,o}$ and

$$J_d(\underline{v}_{br\,\infty}^{\,o}) = J_e(\underline{v}_{r\,\infty}^{\,o}) = \min_{\underline{u}\epsilon U} J_e(\underline{u}) \, .$$

PHYSICAL IMPLEMENTATION OF THE ANALOG NETWORKS

In practice, one has to build networks out of real components and it is hence clear that it will be necessary to resort to special techniques in order to realize a reasonable approximation to the analog networks discussed previously. First, in order to be able to use real transformers, one may use for voltage souces pulse train modulators, with a typical modulator output shown in Fig. 7(a); the corresponding

voltage across the secondary of a pulse transformer is shown in Fig. 7(b) (not to scale). The pulses must be narrow with respect to the period of the pulse train. This is necessary in order to avoid substantial drifts of the "zero" level. One then reads these voltages through a narrow aperture so as to obtain their values at instants when the "transient" ripple has died down. Whenever linear resistors are used, their values must be chosen so as to provide adequate damping to the transformers. (Note that the actual value of these resistors is unimportant.) Current sources may be approximated by placing relatively large resistors in series with the voltage pulse train modulators. Finally, in order to evaluate the effect of nonideal diodes and nonideal Zener diodes, one may assign suitable values to the parameters in the various formulas previously developed.

CONCLUSION

It has been shown in this paper that ideal component circuits can be used to solve minimum fuel and minimum energy problems for linear discrete systems. It has also been shown that effects of using nonideal diodes may be accounted for, while the effect of nonideal transformers can be largely avoided by resorting to pulse techniques. However, as far as a physical implementation is concerned, the requirement of specially wound transformers would make the use of such circuits most practical in the control of mass produced systems. A simple secondary controller would probably have to be added to control the system in a small neighborhood of the desired state in order to ensure asymptotic stability.

The authors have built an elementary optimal system of the kind described in this paper and have found that the network computed controls were within five percent of the required optimal values. As a result, the authors are inclined to encourage the utilization of analog network controllers in simple applications.

FOOTNOTES

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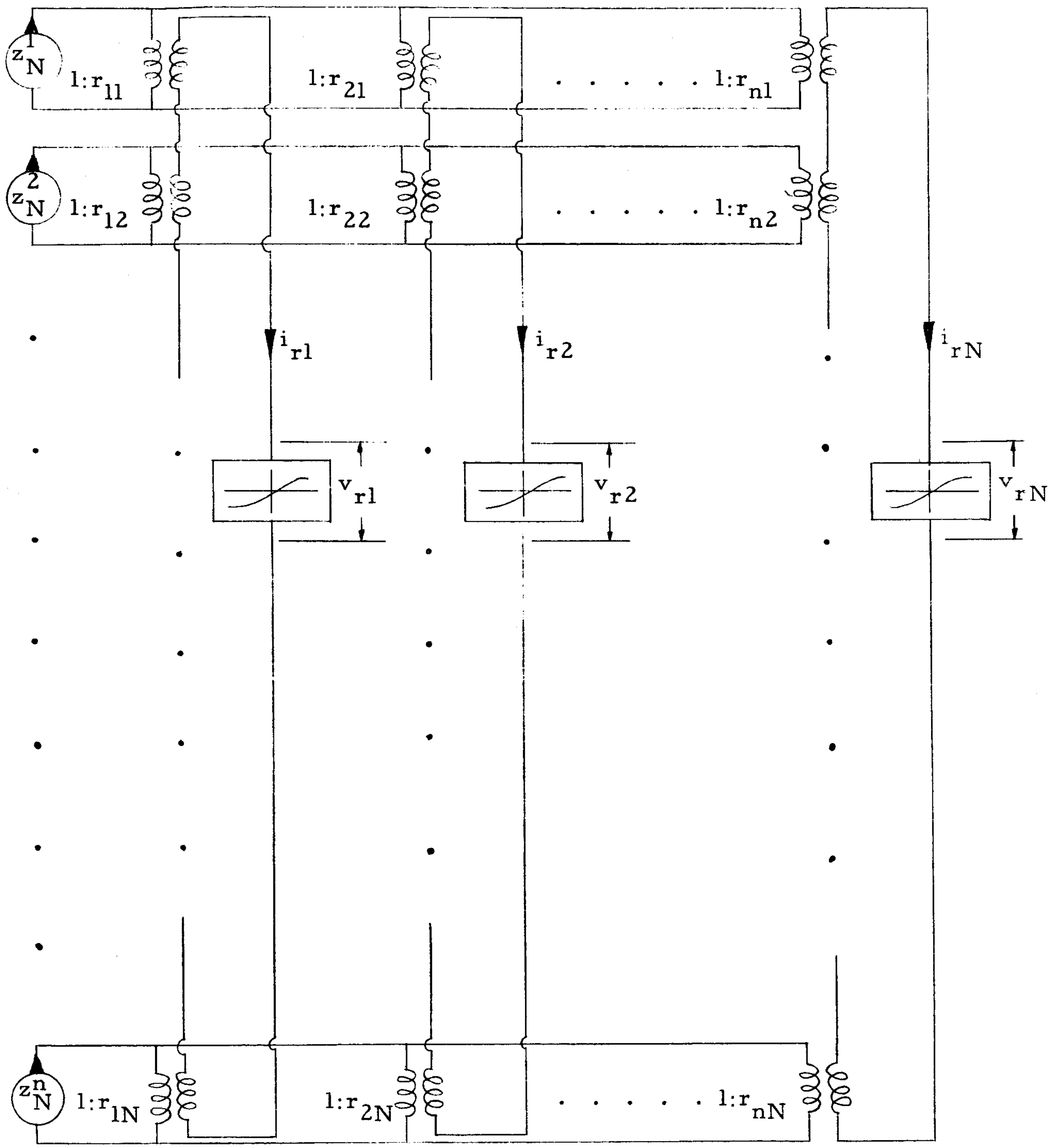

Fig. 1. Current analog for the minimum energy problem.

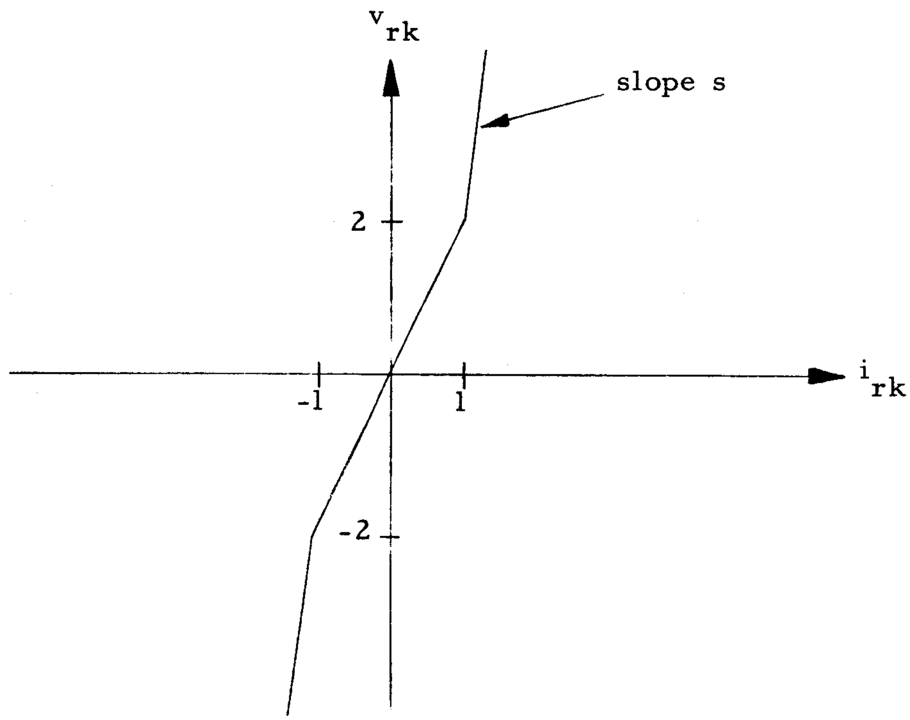

Fig. 2(a). Minimum energy current analog nonlinear resistor characteristic.

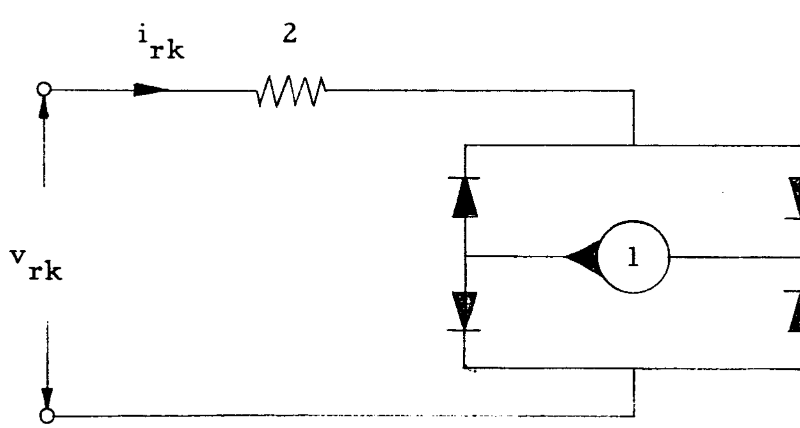

Fig. 2(b). Minimum energy current analog nonlinear resistor realization.

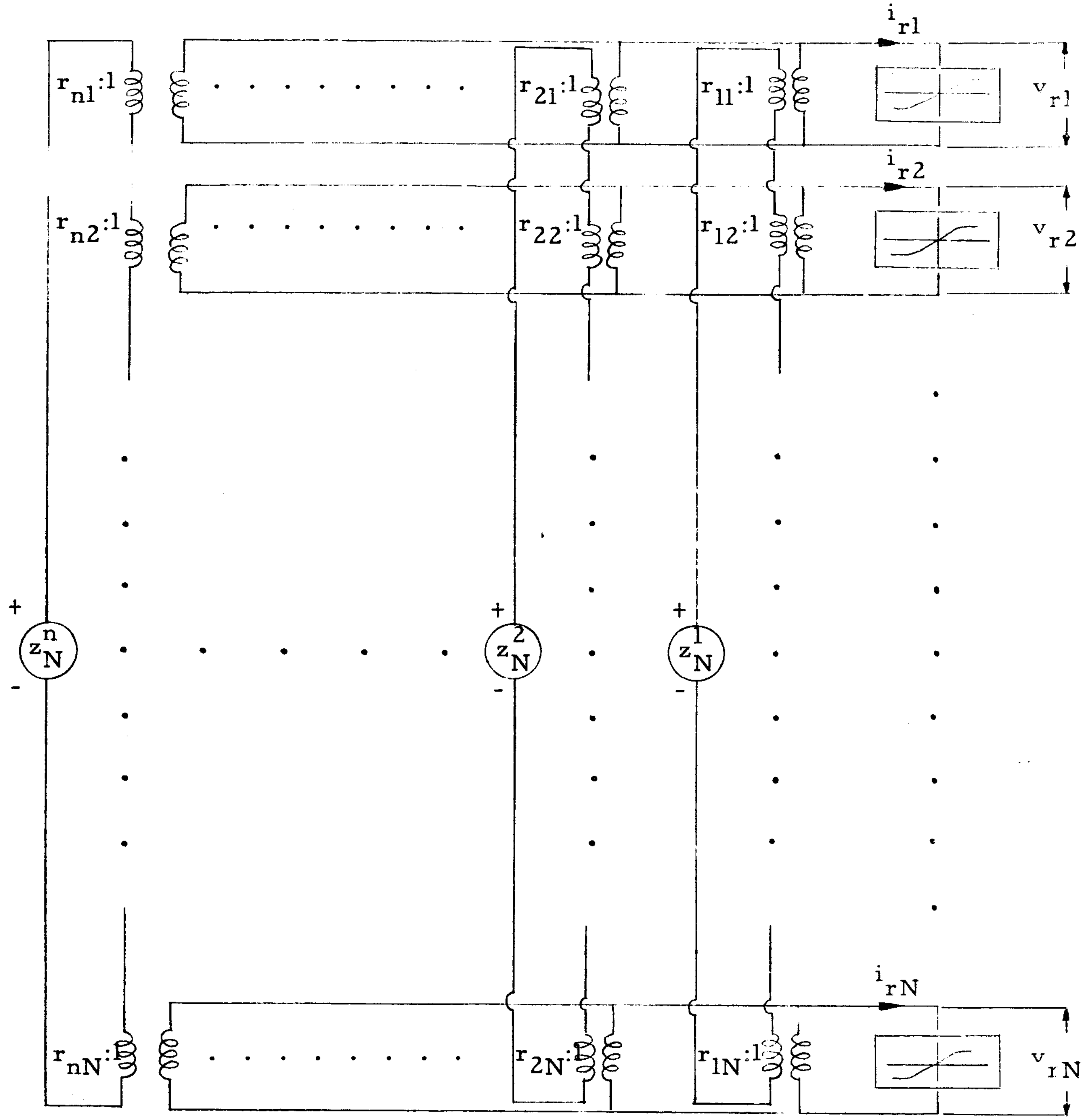

Fig. 3. Voltage analog for the minimum energy problem.

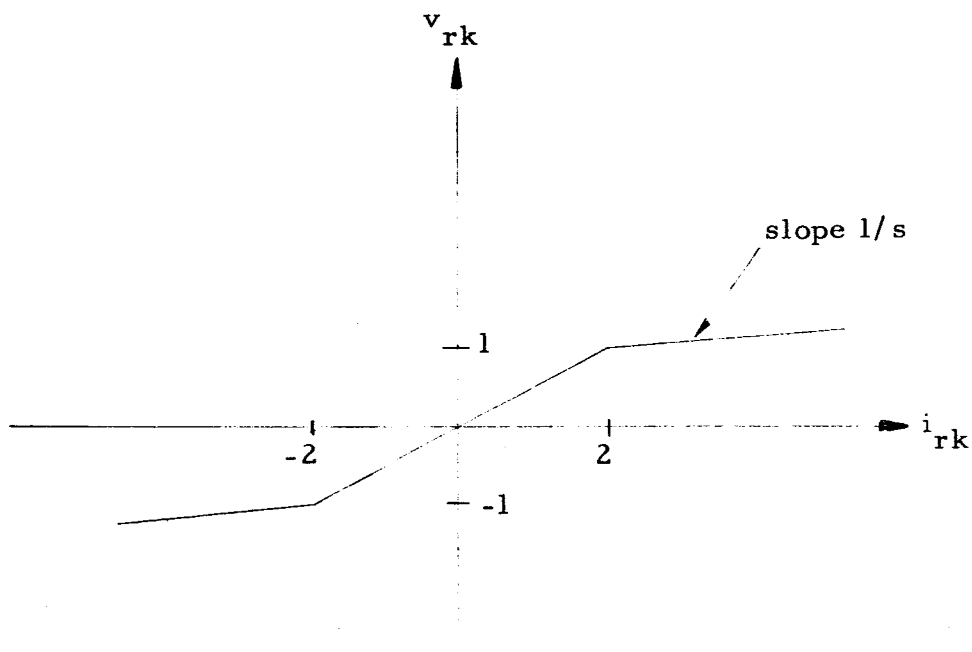

Fig. 4(a). Voltage analog nonlinear resistor characteristic.

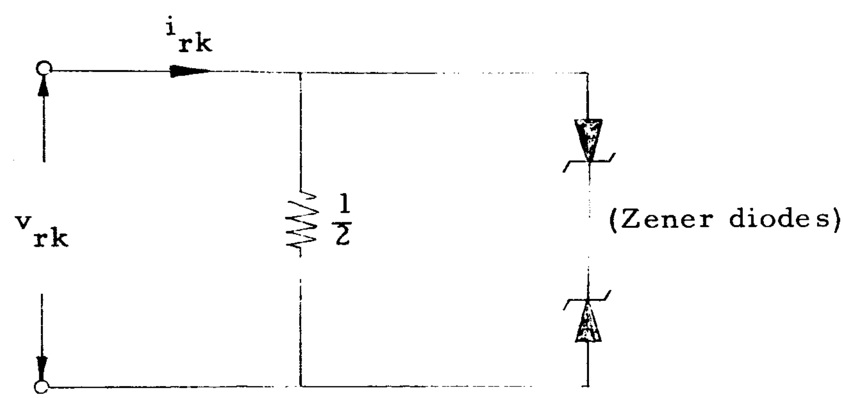

Fig. 4(b). Voltage analog nonlinear resistor realization.

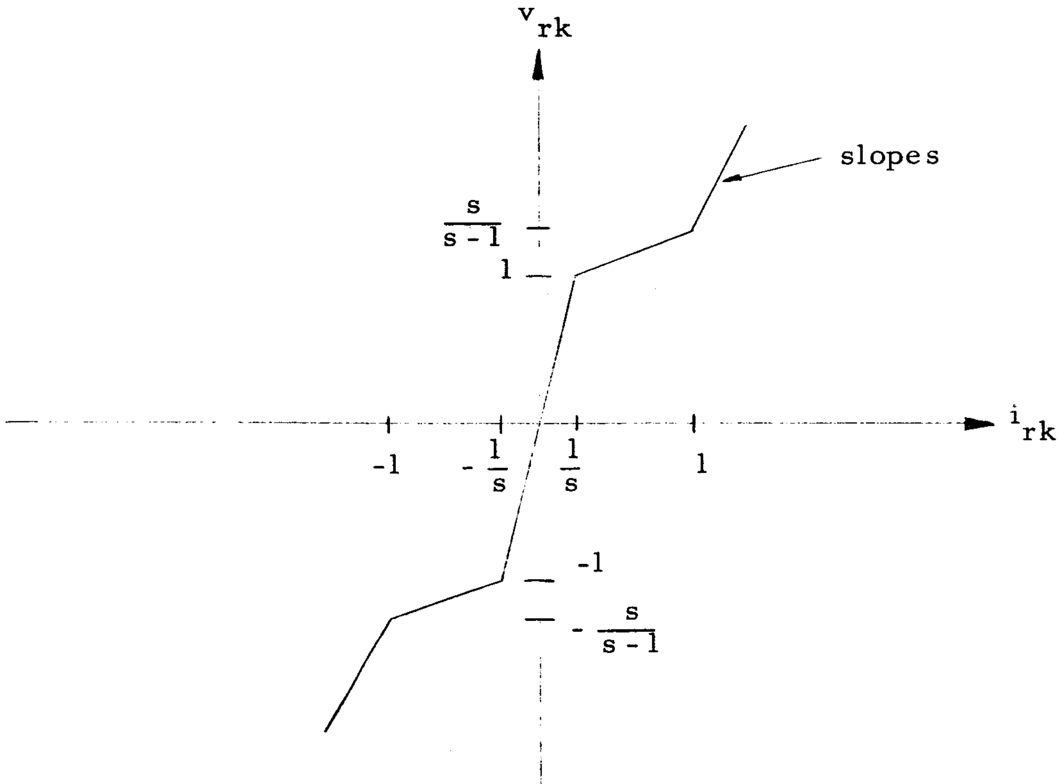

Fig. 5(a). Minimum fuel current analog resistor characteristic.

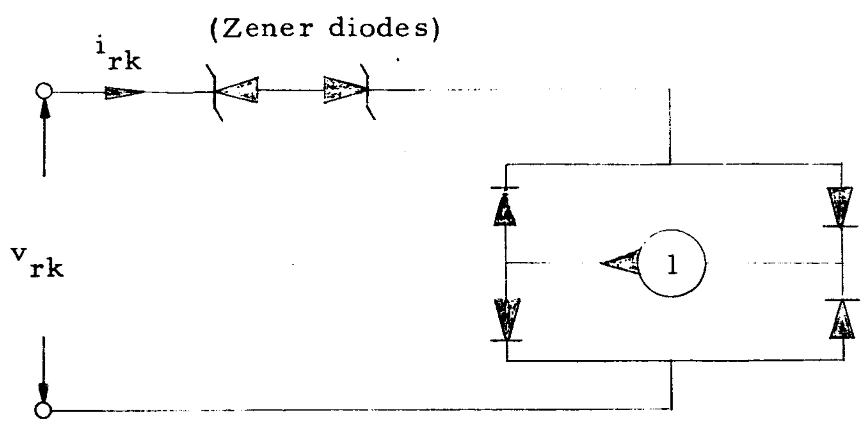

Fig. 5(b). Minimum fuel current analog resistor realization.

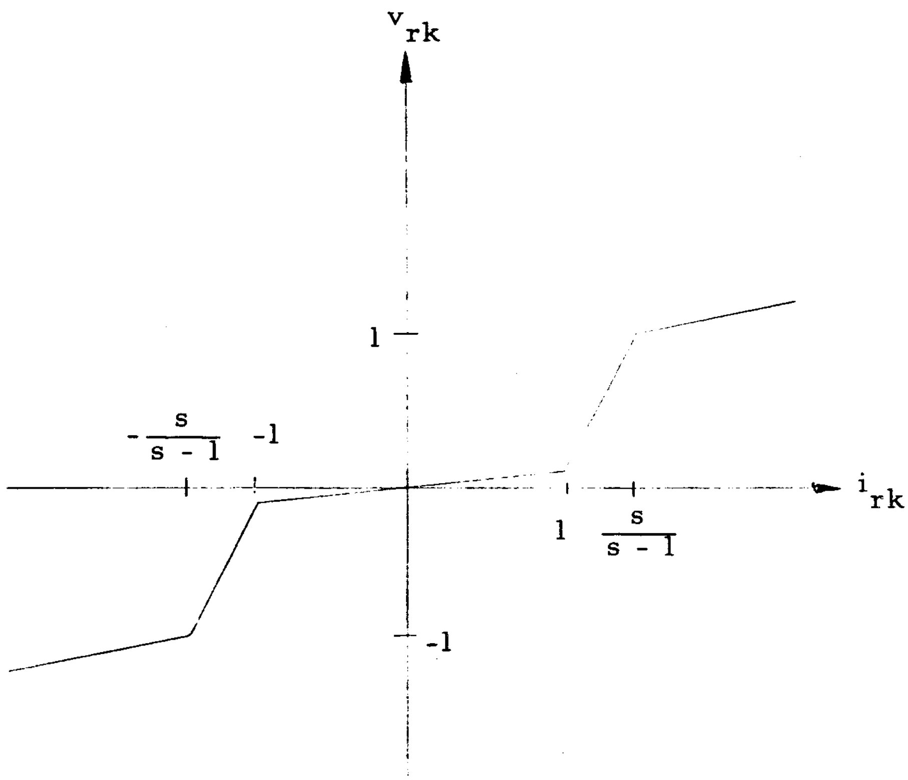

Fig. 6(a). Minimum fuel voltage analog resistor characteristic.

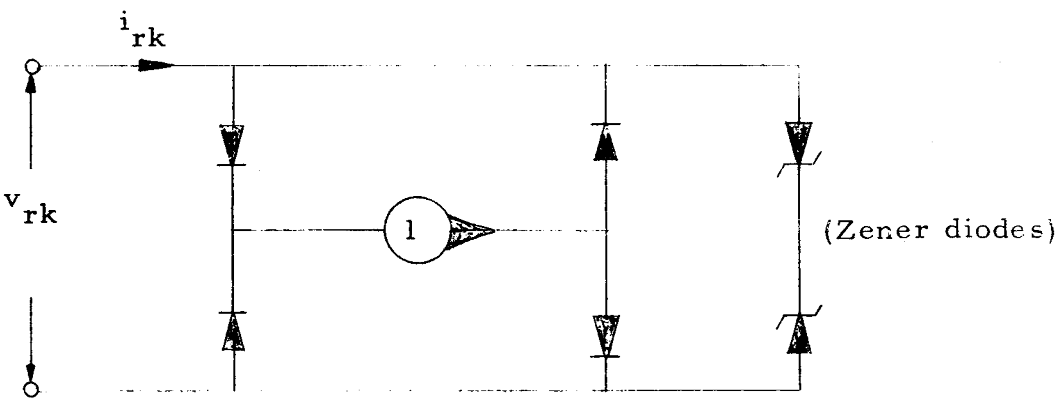

Fig. 6(b). Minimum fuel voltage analog resistor realization.

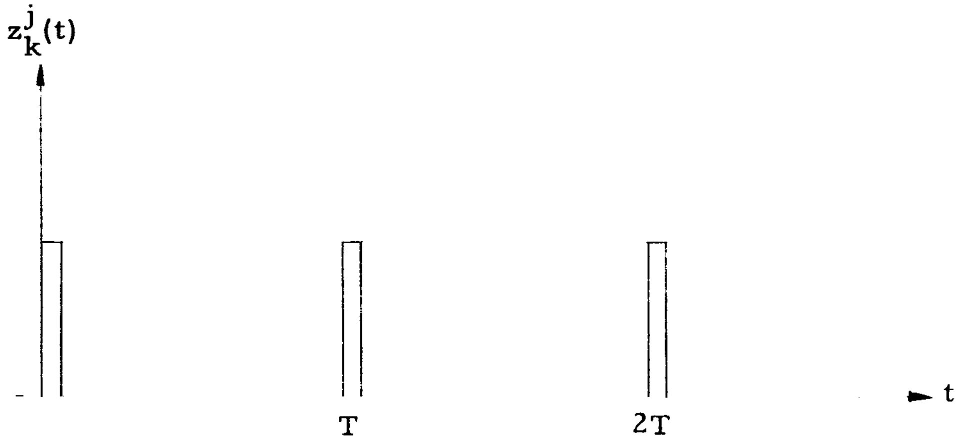

Fig. 7(a). Pulse modulator output.

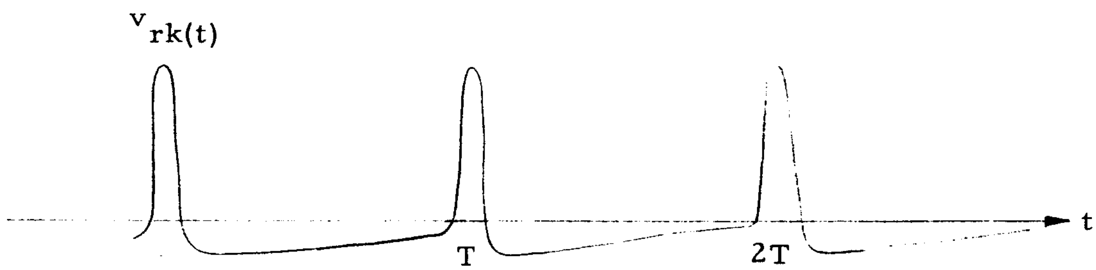

Fig. 7(b). Transformer voltage waveform.